

Approximate multisensor CPHD and PHD filters

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Abstract – *The probability hypothesis density (PHD) filter and cardinalized probability hypothesis density (CPHD) filter are principled approximations of the general multitarget Bayes recursive filter. Both filters are single-sensor filters. Since their multisensor generalizations are computationally intractable, a further approximation—iterating their corrector equations, once for each sensor—has been used instead. This approach is theoretically displeasing because it is not invariant under reordering of the sensors, and because it is implicitly based on strong simplifying assumptions. The purpose of this paper is to derive multisensor PHD and CPHD filters that (1) are invariant under sensor reordering, (2) require much weaker simplifying assumptions, and (3) are potentially computationally tractable (at least in the case of the multisensor CPHD filter).*

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1 Introduction

The probability hypothesis density (PHD) filter [5,7] and cardinalized PHD (CPHD) filter [6,7] are principled approximations of the single-sensor, multitarget recursive Bayes filter. The measurement-update formulas for the PHD and CPHD filters presume the following assumptions about the predicted-target process: it is Poisson (in the case of the PHD filter) or an i.i.d. cluster process (in the case of the CPHD filter).

Given that sensors are independent, it has been known since 2002 that multisensor versions of the PHD and CPHD filters are possible—but also computationally intractable [5]. As in the single-sensor case, these “true” multisensor PHD and CPHD filters are based on the presumption that the predicted-target process is either Poisson or an i.i.d. cluster process. In [3] I verified that the measurement-update formula for the “true” two-sensor PHD filter involves a summation over all binary partitions of the current two-sensor measurement-set.

1.1 The iterated-corrector approximation

Since the “true” multisensor PHD and CPHD filters are computationally intractable, a heuristic approximation—the “iterated-corrector approximation”—has become the default approach for multisensor problems. Consider the PHD filter first. One first uses the corrector step for the

first sensor to derive a measurement-updated PHD for the first sensor. Using this PHD in place of the predicted PHD, one uses the corrector step for the second sensor to derive a measurement-updated PHD for the first two sensors. One iterates this procedure for the remaining sensors. This approach is not entirely satisfactory.

First, changing the order of the sensors produces different measurement-updated PHDs. Simulations thus far indicate that this does not result in noticeable differences in performance. Consequently, the iterated-corrector solution appears to be adequate for practical application. Still, the lack of order-invariance is displeasing from a theoretical point of view.

Second, this approach implicitly requires strong simplifying assumptions. Not only must the predicted-target process be Poisson, but also the successive measurement-updated target processes for each sensor.

The situation is similar for the iterated-corrector CPHD filter. Changing the order of the sensors produces different measurement-updated PHDs and cardinality distributions. Not only must the predicted-target process be an i.i.d. cluster process, but also the successive measurement-updated target processes for each sensor.

1.2 Principled approximation

The purpose of this paper is to demonstrate that there is a middle ground between (on the one hand) the “true” but intractable multisensor filters and (on the other) the tractable but heuristic iterated-corrector filters. I call these two new filters the *product multisensor PHD (PM-PHD) filter* and *product multisensor CPHD (PM-CPHD) filter*. Assume that there are s independent sensors. For both filters, the measurement-update for the PHD has the product form

$$\overset{1..s}{D}_{k+1|k+1}(\mathbf{x}) = K_{Z_{k+1}, \dots, Z_{k+1}}^1 \cdot \overset{1}{L}_{Z_{k+1}}^1(\mathbf{x}) \cdots \overset{s}{L}_{Z_{k+1}}^s(\mathbf{x}) \cdot \overset{1..s}{D}_{k+1|k}(\mathbf{x})$$

where $\overset{j}{L}_{Z_{k+1}}^j(\mathbf{x})$ is the PHD pseudo-likelihood for the j^{th} sensor and where $K_{Z_{k+1}, \dots, Z_{k+1}}^1$ is a constant. Alternatively, this product formula can be written as

$$\overset{1..s}{D}_{k+1|k+1}(\mathbf{x}) = \overset{1..s}{N}_{k+1|k+1} \cdot \frac{\overset{1}{L}_{Z_{k+1}}^1(\mathbf{x}) \cdots \overset{s}{L}_{Z_{k+1}}^s(\mathbf{x}) \cdot \overset{1..s}{S}_{k+1|k}(\mathbf{x})}{\overset{1..s}{S}_{k+1|k}[\overset{1}{L}_{Z_{k+1}}^1 \cdots \overset{s}{L}_{Z_{k+1}}^s]}$$

where $\overset{1..s}{N}_{k+1|k+1}$ is the measurement-updated number of targets, where the normalized predicted PHD is

$$s_{k+1|k}^{1..s}(\mathbf{x}) = \frac{D_{k+1|k}^{1..s}(\mathbf{x})}{N_{k+1|k}^{1..s}}$$

and where, for any function $h(\mathbf{x})$,

$$s_{k+1|k}^{1..s}[h] = \int h(\mathbf{x}) \cdot s_{k+1|k}^{1..s}(\mathbf{x}) d\mathbf{x}$$

In other words, for both the PM-PHD and PM-CPHD filters the measurement-update for the PHD has two steps:

- an update of $N_{k+1|k}^{1..s}$ to $N_{k+1|k+1}^{1..s}$; and
- an update of $s_{k+1|k}^{1..s}(\mathbf{x})$ to $s_{k+1|k+1}^{1..s}(\mathbf{x})$, using the PHD pseudo-likelihoods as though they are actual likelihoods in the following Bayes' rule-like formula:

$$s_{k+1|k+1}^{1..s}(\mathbf{x}) = \frac{L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x}) \cdot s_{k+1|k}^{1..s}(\mathbf{x})}{s_{k+1|k}^{1..s}[L_{k+1}^1 \cdots L_{k+1}^s]}$$

1.3 The new approximations

What new approximation techniques lead to these new filters? Assuming that s sensors are independent, the multisensor measurement-update step for the multitarget Bayes filter at time k is

$$f_{k|k}(X) \propto f_k^1(Z_k | X) \cdots f_k^s(Z_k | X) \cdot f_{k|k-1}(X) \quad (1)$$

where $f_{k|k-1}(X)$ is the predicted multitarget distribution, where Z_k^j is the measurement-set collected by the j^{th} sensor, and where $f_k^j(Z_k^j | X)$ is the multitarget likelihood function for the j^{th} sensor. Eq. (1) can be rewritten as

$$f_{k|k}(X) \propto f_{k|k}(X | Z_k^1) \cdots f_{k|k}(X | Z_k^s) \cdot f_{k|k-1}(X)^{-s+1} \quad (2)$$

where

$$f_{k|k}(X | Z_k^j) \propto f_k^j(Z_k^j | X) \cdot f_{k|k-1}(X) \quad (3)$$

is the measurement-updated multitarget distribution, but updated using only the j^{th} sensor. (Eq. (2) is known as “Bayes parallel combination” [7, p. 137].)

The new multisensor PHD and CPHD filters are based on the following simplifying assumptions:

- For the PM-CPHD filter: $f_{k|k-1}(X)$ and the $f_{k|k}(X | Z_k^j)$ are the distributions of i.i.d. cluster processes, and the sensor clutter processes are i.i.d. cluster processes.
- For the PM-PHD filter: $f_{k|k-1}(X)$ is Poisson, the $f_{k|k}(X | Z_k^j)$ are the distributions of i.i.d. cluster processes, and the sensor clutter processes are Poisson. Given these assumptions, the new filters are derived directly from Eq. (2) by substitution.

Caution: In the case of the PM-PHD filter, one might be tempted to impose a stronger assumption: that the $f_{k|k}(X | Z_k^j)$ are the distributions of Poisson processes rather than i.i.d. cluster processes. Doing so would lead to a multisensor PHD filter with poor performance. Specifically, it would lead to the following corrector:

$$D_{k+1|k+1}^{1..s}(\mathbf{x}) = L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x}) \cdot D_{k+1|k}^{1..s}(\mathbf{x})$$

Under single-target assumptions (no missed detections, no false alarms, and target number is known *a priori* to be 1), this equation does not reduce to the corrector for the multisensor, single-target Bayes filter—i.e., to Bayes' rule.

1.4 Characteristics of the new filters

The corrector formulas of the PM-PHD and PM-CPHD filters have the following characteristics.

1. The PM-CPHD filter equations appear to be potentially computationally tractable. The same is not necessarily true for the PM-PHD filter equations. See Section 4.
2. When all sensors fail to collect measurements, the PM-PHD filter reduces to the formula provided by the “true” multisensor PHD filter described in [3].
3. In the single-sensor case, the PM-CPHD filter reduces to the CPHD filter and the PM-PHD filter reduces to the PHD filter.
4. Suppose that there are no missed detections, no false alarms, and that the number of targets is known *a priori* to be one—that is, assume the single-target filtering scenario. Then the PM-CPHD and PM-PHD filters both reduce to the multisensor, single-target Bayes filter.
5. Both filters exhibit the same “spooky action at a distance” behavior that Fränken and Ulmke [1] identified in the single-sensor CPHD filter. This means that multitarget scenarios should be broken up into statistically non-interacting clusters, and a separate filter assigned to each cluster.
6. Because of this “spookiness,” the PM-CPHD and PM-PHD filters do not behave intuitively when the sensors have mutually disjoint fields of view. These filters do not reduce to separate, non-interacting filters, one for each field of view (as is the case with the “true” multisensor PHD filter described in [3]).

It should be mentioned in this context that a so-called “general multisensor intensity filter” has been claimed to be superior to the iterated-corrector PHD filter. However, I have demonstrated [4] that this “filter” is profoundly erroneous. It does not reduce to the correct (and intuitively obvious) answers in important special cases—e.g., cases 2 and 4 just mentioned—and will perform badly in even the easiest multitarget tracking problems.

The main reason is that the measurement-update equation of the “intensity filter” does not involve a product $L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x})$ of pseudo-likelihoods—as one would intuitively expect, given that the sensors are assumed independent. Rather, it involves a sum $L_{k+1}^1(\mathbf{x}) + \dots + L_{k+1}^s(\mathbf{x})$ of pseudo-likelihoods—which is clearly not compatible with sensor independence.

(The *single-sensor* “intensity filter” is also seriously erroneous, but for different reasons. This question is addressed in Section 6 of the companion paper [2])

1.5 Organization of the paper

The paper is organized as follows. Section 2 summarizes the multisensor CPHD filter and its special case, the multisensor PHD filter. The mathematical derivation of these filters is in Section 3. A summary and conclusions can be found in Section 4.

Note 1: Those readers who have seen the 1984 Miloš Forman movie *Amadeus*, may recall the scene in which Emperor Joseph II objects that Mozart's new orchestral composition "has too many notes." Astonished, Mozart responds, "there are just as many notes as are required, Majesty, neither more nor less."

I suggest that so it is with this paper. To my astonishment, a reviewer has objected that it *has too many equations*. My response is that *there are just as many equations as are necessary*. The paper contains many equations because the derivation of its main result is based on a systematic, careful, and mathematically rigorous methodology—as opposed, say, to the approach critiqued in [4], which has far fewer equations, at the price of also being profoundly erroneous. The number of equations in this paper could be reduced, but only at the expense of rendering the derivation less transparent to those readers who are inclined to follow it step-by-step to verify—or challenge—its correctness. Those readers who are not so inclined, are encouraged to skip as many equations as they wish.

Note 2: This reviewer also objected to the fact that the paper contains no simulations. The purpose of this paper is to show that a "middle ground" approximation of the "true" multisensor PHD and CPHD filters can be achieved, and to provide a systematic, mathematically rigorous, and sufficiently detailed mathematical demonstration of this fact. Given the conference page limit, it would have been impossible to also include simulations. Implementation results will be the subject of some future paper.

2 Main results

I summarize the PM-CPHD filter in Section 2.1 and the PM-PHD filter in Section 2.2.

In what follows, $\sigma_{m,i}(x_1, \dots, x_m)$ is the elementary symmetric function of degree i in m variables x_1, \dots, x_m . For any function $f(x)$, $f^{(i)}(x)$ denotes the i^{th} derivative of $f(x)$. Given any probability distribution $p(n)$,

$$G(x) = \sum_{n \geq 0} p(n) \cdot x^n$$

is the *probability generating function* (p.g.f.) of $p(n)$.

2.1 The multisensor CPHD (PM-CPHD) filter

Suppose that there are s independent sensors and that, for $j = 1, \dots, s$, the j^{th} sensor has collected the sequence $Z^{(k)} : Z_1^j, \dots, Z_k^j$ of measurement-sets. Suppose that these

measurement-sets have resulted in the following multisensor PHD and multisensor cardinality distribution:

$$\overset{1..s}{D}_{k|k}(\mathbf{x}) = \overset{abbr.}{D}_{k|k}(\mathbf{x} | Z^{(k)}, \dots, Z^{(k)}) \quad (4)$$

$$\overset{1..s}{p}_{k|k}(n) = \overset{abbr.}{p}_{k|k}(n | Z^{(k)}, \dots, Z^{(k)}) \quad (5)$$

Suppose that the usual CPHD filter time-update equations have been used to construct the predicted PHD $\overset{1..s}{D}_{k+1|k}(\mathbf{x})$ and predicted cardinality distribution $\overset{1..s}{p}_{k+1|k}(n)$. Let

$$\overset{1..s}{N}_{k+1|k} = \int \overset{1..s}{D}_{k+1|k}(\mathbf{x}) d\mathbf{x}, \quad \overset{1..s}{s}_{k+1|k}(\mathbf{x}) = \frac{\overset{1..s}{D}_{k+1|k}(\mathbf{x})}{\overset{1..s}{N}_{k+1|k}} \quad (6)$$

and let $\overset{1..s}{G}_{k+1|k}(x)$ be the p.g.f. of $\overset{1..s}{p}_{k+1|k}(n)$.

For $j = 1, \dots, s$, let the probability of detection and likelihood function of the j^{th} sensor be

$$p_D^j(\mathbf{x}), \quad L_z^j(\mathbf{x}) = f_{k+1}^j(\mathbf{z} | \mathbf{x}) \quad (7)$$

Also, the clutter process for each sensor is assumed to be an i.i.d. clutter process. Let

$$c_{k+1}^j(\mathbf{z}), \quad C_{k+1}^j(z) \quad (8)$$

be the clutter spatial distribution and probability generating function (p.g.f.) of the clutter process.

Finally suppose that, at time-step $k+1$ and for $j = 1, \dots, s$, the sensors simultaneously collect new measurement-sets $Z_{k+1}^1, \dots, Z_{k+1}^s$ with $|Z_{k+1}^j| = m$. Then in Section 4.2 I show that the measurement-updated multisensor PHD and cardinality distribution are

$$\overset{1..s}{D}_{k+1|k+1}(\mathbf{x}) = \overset{1..s}{L}_{Z_{k+1}^1, \dots, Z_{k+1}^s}(\mathbf{x}) \cdot \overset{1..s}{s}_{k+1|k}(\mathbf{x}) \quad (9)$$

$$\overset{1..s}{p}_{k+1|k+1}(n) = \frac{\tilde{p}(n) \cdot \sigma^n}{\tilde{G}(\sigma)} \quad (10)$$

where

$$\overset{1..s}{L}_{Z_{k+1}^1, \dots, Z_{k+1}^s}(\mathbf{x}) = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \cdot \frac{\overset{1}{L}_{Z_{k+1}^1}(\mathbf{x}) \cdots \overset{s}{L}_{Z_{k+1}^s}(\mathbf{x})}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \quad (11)$$

$$\sigma = \frac{\overset{1..s}{s}_{k+1|k}[\overset{1}{L}_{Z_{k+1}^1} \cdots \overset{s}{L}_{Z_{k+1}^s}]}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \quad (12)$$

$$\tilde{G}(x) = \sum_{n=0}^{\infty} \tilde{p}(n) \cdot x^n \quad (13)$$

$$\tilde{p}(n) = \overset{1}{\ell}_{Z_{k+1}^1}(n) \cdots \overset{s}{\ell}_{Z_{k+1}^s}(n) \cdot \overset{1..s}{p}_{k+1|k}(n) \quad (14)$$

and where, for $j = 1, \dots, s$,

$$\overset{j}{\ell}_{Z_{k+1}^j}(n) = \sum_{l=0}^{\min(n, m)} \overset{j}{C}_{k+1}^{(m-l)}(0) \cdot l! \cdot \overset{1..s}{s}_{k+1|k} [1 - p_D^j]^{n-l} \cdot \overset{j}{\sigma}_l(Z_{k+1}^j) \quad (15)$$

$$\overset{j}{L}_{Z_{k+1}^j}(\mathbf{x}) = \overset{j}{\alpha}_0 (1 - p_D^j(\mathbf{x})) + \sum_{\mathbf{z} \in Z_{k+1}^j} \frac{\overset{j}{p}_D(\mathbf{x}) \cdot \overset{j}{L}_z(\mathbf{x}) \cdot \overset{j}{\alpha}(\mathbf{z})}{\overset{j}{c}_{k+1}(\mathbf{z})} \quad (16)$$

$$\overset{j}{N}_{k+1|k+1} = \overset{j}{\alpha}_0 \overset{1..s}{s}_{k+1|k} [1 - p_D^j] + \sum_{\mathbf{z} \in Z_{k+1}^j} \frac{\overset{1..s}{s}_{k+1|k} [p_D^j \overset{j}{L}_z] \cdot \overset{j}{\alpha}(\mathbf{z})}{\overset{j}{c}_{k+1}(\mathbf{z})} \quad (17)$$

$$\alpha_0 = \frac{\sum_{l=0}^j C_{k+1}^{(m-l)}(0) \cdot G_{k+1|k}^{(l+1)}(\gamma) \cdot \sigma_l(Z_{k+1})}{\sum_{i=0}^j C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(\gamma) \cdot \sigma_i(Z_{k+1})} \quad (18)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^{m-1} C_{k+1}^{(m-l-1)}(0) \cdot G_{k+1|k}^{(l+1)}(\gamma) \cdot \sigma_l(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^j C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(\gamma) \cdot \sigma_i(Z_{k+1})} \quad (19)$$

$$\sigma_i(Z_{k+1}) = \sigma_{m,j} \left(\frac{s_{k+1|k}[p_D L_{z_1}]^{1..s}}{c_{k+1}(z_1)}, \dots, \frac{s_{k+1|k}[p_D L_{z_m}]^{1..s}}{c_{k+1}(z_m)} \right) \quad (20)$$

$$\gamma = s_{k+1|k}[1 - p_D] \quad (21)$$

2.2 The multisensor PHD (PM-PHD) filter

Beyond the assumptions made in Section 2.1, assume that the sensor clutter processes and the predicted target processes are Poisson. Let λ_{k+1}^j be the clutter rate for the j^{th} sensor. Then in Section 4.3 I show that the PM-PHD measurement-update equation is

$$D_{k+1|k+1}(\mathbf{x}) = L_{z_{k+1}, \dots, z_{k+1}}^{1..s}(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) \quad (22)$$

where

$$L_{z_{k+1}, \dots, z_{k+1}}^{1..s}(\mathbf{x}) = \phi \cdot \frac{L_{z_{k+1}}^1(\mathbf{x}) \cdots L_{z_{k+1}}^s(\mathbf{x})}{V_{k+1|k+1} \cdots V_{k+1|k+1}} \quad (23)$$

$$\sigma = \frac{s_{k+1|k}[L_{z_{k+1}}^1 \cdots L_{z_{k+1}}^s]}{V_{k+1|k+1} \cdots V_{k+1|k+1}} \quad (24)$$

$$\phi = \frac{\sum_{n \geq 0} \ell_{z_{k+1}}^1(n+1) \cdots \ell_{z_{k+1}}^s(n+1) \cdot \frac{(N_{k+1|k} \cdot \sigma)^n}{n!}}{\sum_{j \geq 0} \ell_{z_{k+1}}^1(j) \cdots \ell_{z_{k+1}}^s(j) \cdot \frac{(N_{k+1|k} \cdot \sigma)^j}{j!}} \quad (25)$$

and where for $j = 1, \dots, s$

$$\ell_{z_{k+1}}^j(n) = \sum_{l=0}^{\min\{n, m\}} \lambda_{k+1}^{m-l} \cdot j! \cdot C_{n,l} \cdot s_{k+1|k}[1 - p_D]^{n-l} \cdot \sigma_l(Z_{k+1}) \quad (26)$$

$$L_{z_{k+1}}^j(\mathbf{x}) = 1 - p_D(\mathbf{x}) + \sum_{z \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_z^j(\mathbf{x})}{\lambda_{k+1} c_{k+1}(z) + D_{k+1|k}[p_D L_z^j]} \quad (27)$$

$$V_{k+1|k+1} = s_{k+1|k}[1 - p_D] + \sum_{z \in Z_{k+1}} \frac{s_{k+1|k}[p_D L_z^j]}{\lambda_{k+1} c_{k+1}(z) + D_{k+1|k}[p_D L_z^j]} \quad (28)$$

$$\sigma_i(Z_{k+1}) = \sigma_{m,j} \left(\frac{s_{k+1|k}[p_D L_{z_1}]^{1..s}}{c_{k+1}(z_1)}, \dots, \frac{s_{k+1|k}[p_D L_{z_m}]^{1..s}}{c_{k+1}(z_m)} \right) \quad (29)$$

3 The single-sensor CPHD filter (review)

In this section I briefly summarize the measurement-update equations for the single-sensor CPHD filter. Suppose that the usual CPHD filter time-update equations have been used to construct the predicted PHD $D_{k+1|k}(\mathbf{x})$ and predicted cardinality distribution $p_{k+1|k}(n)$. Let

$$N_{k+1|k} = \int D_{k+1|k}(\mathbf{x}) d\mathbf{x}, \quad s_{k+1|k}(\mathbf{x}) = \frac{D_{k+1|k}(\mathbf{x})}{N_{k+1|k}} \quad (30)$$

and let $G_{k+1|k}(x)$ be the p.g.f. of $p_{k+1|k}(n)$. Let the likelihood function and probability of detection of the sensor be $L_z(\mathbf{x}) = f_{k+1}(z | \mathbf{x})$ and $p_D(\mathbf{x})$. The sensor clutter process is an i.i.d. clutter process where $c_{k+1}(z)$ is the clutter spatial distribution and where $C_{k+1}(z)$ is the p.g.f. of the clutter process. Suppose that, at time-step $k+1$ the sensor collects a new measurement-set Z_{k+1} with $|Z_{k+1}| = m$. Then the measurement-updated PHD and cardinality distribution are

$$D_{k+1|k+1}(\mathbf{x}) = L_{Z_{k+1}}(\mathbf{x}) \cdot s_{k+1|k}(\mathbf{x}) \quad (31)$$

$$p_{k+1|k+1}(n) = \frac{\ell_{Z_{k+1}}(n) \cdot p_{k+1|k}(n)}{\sum_{j \geq 0} \ell_{Z_{k+1}}(j) \cdot p_{k+1|k}(j)} \quad (32)$$

where

$$L_{Z_{k+1}}(\mathbf{x}) = \alpha_0(1 - p_D(\mathbf{x})) + \sum_{z \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_z(\mathbf{x}) \cdot \alpha(\mathbf{z})}{c_{k+1}(z)} \quad (33)$$

$$\ell_z(n) = \sum_{l=0}^{\min\{n, m\}} C_{k+1}^{(m-l)}(0) \cdot j! \cdot C_{n,l} \cdot s_{k+1|k}[1 - p_D]^{n-l} \cdot \sigma_l(Z) \quad (34)$$

$$\alpha_0 = \frac{\sum_{l=0}^m C_{k+1}^{(m-l)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}[1 - p_D]) \cdot \sigma_l(Z_{k+1})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}[1 - p_D]) \cdot \sigma_i(Z_{k+1})} \quad (35)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^{m-1} C_{k+1}^{(m-l-1)}(0) G_{k+1|k}^{(l+1)}(s_{k+1|k}[1 - p_D]) \cdot \sigma_l(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}[1 - p_D]) \cdot \sigma_i(Z_{k+1})} \quad (36)$$

$$\sigma_i(Z_{k+1}) = \sigma_{m,j} \left(\frac{s_{k+1|k}[p_D L_{z_1}]^{1..s}}{c_{k+1}(z_1)}, \dots, \frac{s_{k+1|k}[p_D L_{z_m}]^{1..s}}{c_{k+1}(z_m)} \right) \quad (37)$$

4 Mathematical derivations

The derivations of the PM-CPHD and PM-PHD filters follow. Without loss of generality I can assume that there are $s = 2$ sensors. In this case, Eq. (2) becomes

$$f_{k+1|k+1}(X) \propto f_{k+1|k+1}(X | Z_k) \cdot f_{k+1|k+1}(X | Z_k^2) \cdot f_{k+1|k}(X)^{-1} \quad (38)$$

In what follows I prove a Lemma (Section 4.1), derive the PM-CPHD filter measurement-update equations (Section 4.2), and derive the PM-PHD filter measurement-update equations (Section 4.3)

4.1 Lemma: Bayes parallel combination for i.i.d. cluster processes

I use the following notation in what follows. Let $h(\mathbf{x})$ be a real-valued function and X a finite set. Then define

$$h^X = \begin{cases} 1 & \text{if } X = \emptyset \\ \prod_{\mathbf{x} \in X} h(\mathbf{x}) & \text{if } X \neq \emptyset \end{cases} \quad (39)$$

In Eq. (38), assume that all three multitarget densities on the right are the distributions of i.i.d. cluster processes:

$$f_{k+1|k+1}(X | Z_k) = |X|! \cdot p_{k+1|k+1}^1(n) \cdot s_{k+1|k+1}^X \quad (40)$$

$$f_{k+1|k+1}(X | Z_k) = |X|! \cdot p_{k+1|k+1}^2(n) \cdot s_{k+1|k+1}^X \quad (41)$$

$$f_{k+1|k}(X) = |X|! \cdot p_{k+1|k}^{12}(n) \cdot s_{k+1|k}^X \quad (42)$$

Then I show that

$$f_{k+1|k+1}(X) = |X|! \cdot p_{k+1|k+1}^{12}(n) \cdot s_{k+1|k+1}^X \quad (43)$$

where

$$p_{k+1|k+1}^{12}(n) = \frac{p(n) \cdot \sigma^n}{G(\sigma)} \quad (44)$$

$$s_{k+1|k+1}^{12}(\mathbf{x}) = \frac{G^{(1)}(\sigma)}{G(\sigma)} \cdot \frac{s_{k+1|k+1}^1(\mathbf{x}) \cdot s_{k+1|k+1}^2(\mathbf{x}) \cdot s_{k+1|k+1}^{12}(\mathbf{x})^{-1}}{\sigma} \quad (45)$$

and where

$$\sigma = \int s_{k+1|k+1}^1(\mathbf{x}) \cdot s_{k+1|k+1}^2(\mathbf{x}) \cdot s_{k+1|k+1}^{12}(\mathbf{x})^{-1} d\mathbf{x} \quad (46)$$

$$p(n) = \frac{p_{k+1|k+1}^1(n) \cdot p_{k+1|k+1}^2(n) \cdot p_{k+1|k}^{12}(n)^{-1}}{\sum_{j \geq 0} p_{k+1|k+1}^1(j) \cdot p_{k+1|k+1}^2(j) \cdot p_{k+1|k}^{12}(j)^{-1}} \quad (47)$$

$$G(x) = \sum_{n \geq 0} p(n) \cdot x^n \quad (48)$$

For, substituting Eqs. (40-42) into Eq. (38) we get

$$f_{k+1|k+1}(X) \propto |X|! \cdot p_{k+1|k+1}^1(|X|) \cdot p_{k+1|k+1}^2(|X|) \cdot p_{k+1|k}^{12}(|X|)^{-1} \cdot (s_{k+1|k+1}^1 s_{k+1|k+1}^2 s_{k+1|k}^{12})^X \quad (49)$$

Integrating the right-hand side using a set integral yields

$$K = \sum_{n \geq 0} p_{k+1|k+1}^1(n) \cdot p_{k+1|k+1}^2(n) \cdot p_{k+1|k}^{12}(n)^{-1} \cdot \sigma^n \quad (50)$$

from which the result immediately follows.

4.2 Derivation of PM-CPHD filter equations

The multisensor CPHD filter equations that were summarized in Section 2.1 are an immediate consequence of Eqs. (44,45) and the measurement-update equations for the usual CPHD filter, Eqs. (30-37). First, from Eq. (32),

$$p_{k+1|k+1}^1(n) = \frac{\ell_{Z_{k+1}}^1(n) \cdot p_{k+1|k}^{12}(n)}{\sum_{j \geq 0} \ell_{Z_{k+1}}^1(j) \cdot p_{k+1|k}^{12}(j)} \quad (51)$$

$$p_{k+1|k+1}^2(n) = \frac{\ell_{Z_{k+1}}^2(n) \cdot p_{k+1|k}^{12}(n)}{\sum_{j \geq 0} \ell_{Z_{k+1}}^2(j) \cdot p_{k+1|k}^{12}(j)} \quad (52)$$

where, from Eq. (34),

$$\ell_{Z_{k+1}}^1(n) = \sum_{l=0}^{\min\{n,m\}} C_{k+1}^{(m-l)}(0) \cdot j! \cdot C_{n,l}^{12} \cdot s_{k+1|k}^{12} [1 - p_D]^{n-l} \cdot \sigma_l^1(Z) \quad (53)$$

$$\ell_{Z_{k+1}}^2(n) = \sum_{l=0}^{\min\{n,m\}} C_{k+1}^{(m-l)}(0) \cdot j! \cdot C_{n,l}^{12} \cdot s_{k+1|k}^{12} [1 - p_D]^{n-l} \cdot \sigma_l^2(Z) \quad (54)$$

Substituting Eqs. (51,52) into Eq. (47) yields

$$p(n) = \frac{\tilde{p}(n)}{\sum_{j \geq 0} \tilde{p}(j)} \quad (55)$$

where

$$\tilde{p}(n) = \ell_{Z_{k+1}}^1(n) \cdot \ell_{Z_{k+1}}^2(n) \cdot p_{k+1|k}^{12}(n) \quad (56)$$

Second, from Eq. (31) we can write

$$D_{k+1|k+1}^1(\mathbf{x}) = L_{Z_{k+1}}^1(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x}) \quad (57)$$

$$D_{k+1|k+1}^2(\mathbf{x}) = L_{Z_{k+1}}^2(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x}) \quad (58)$$

or, equivalently,

$$s_{k+1|k+1}^1(\mathbf{x}) = \frac{L_{Z_{k+1}}^1(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x})}{N_{k+1|k+1}^1} \quad (59)$$

$$s_{k+1|k+1}^2(\mathbf{x}) = \frac{L_{Z_{k+1}}^2(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x})}{N_{k+1|k+1}^2} \quad (60)$$

where

$$N_{k+1|k+1}^1 = \int L_{Z_{k+1}}^1(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x}) d\mathbf{x} \quad (61)$$

$$N_{k+1|k+1}^2 = \int L_{Z_{k+1}}^2(\mathbf{x}) \cdot s_{k+1|k}^{12}(\mathbf{x}) d\mathbf{x} \quad (62)$$

and where, from Eq. (33),

$$L_{Z_{k+1}}^1(\mathbf{x}) = \alpha_0(1 - p_D(\mathbf{x})) + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_z^1(\mathbf{x}) \cdot \alpha(\mathbf{z})}{c_{k+1}^1(\mathbf{z})} \quad (63)$$

$$L_{Z_{k+1}}^2(\mathbf{x}) = \alpha_0(1 - p_D(\mathbf{x})) + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_z^2(\mathbf{x}) \cdot \alpha(\mathbf{z})}{c_{k+1}^2(\mathbf{z})} \quad (64)$$

and where, from Eqs. (35,36)

$$\alpha_0 = \frac{\sum_{l=0}^m C_{k+1}^{(m-l)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^1(Z_{k+1})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^1(Z_{k+1})} \quad (65)$$

$$\alpha_0 = \frac{\sum_{l=0}^m C_{k+1}^{(m-l)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^2(Z_{k+1})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^2(Z_{k+1})} \quad (66)$$

$$\alpha_0 = \frac{\sum_{l=0}^m C_{k+1}^{(m-l)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^1(Z_{k+1})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^1(Z_{k+1})} \quad (67)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^m C_{k+1}^{(m-l-1)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^1(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^1(Z_{k+1})} \quad (68)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^m C_{k+1}^{(m-l-1)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^2(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^2(Z_{k+1})} \quad (68)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^m C_{k+1}^{(m-l-1)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^1(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^1(Z_{k+1})} \quad (68)$$

$$\alpha(\mathbf{z}) = \frac{\sum_{l=0}^m C_{k+1}^{(m-l-1)}(0) \cdot G_{k+1|k}^{(l+1)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_l^2(Z_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^m C_{k+1}^{(m-i)}(0) \cdot G_{k+1|k}^{(i)}(s_{k+1|k}^{12} [1 - p_D]) \cdot \sigma_i^2(Z_{k+1})} \quad (68)$$

Thus it follows that

$$\overset{1}{N}_{k+1|k+1} = \overset{1}{\alpha}_0 \overset{12}{S}_{k+1|k} [1 - \overset{1}{p}_D] + \sum_{\overset{1}{\mathbf{z}} \in \overset{1}{Z}_{k+1}} \frac{\overset{12}{S}_{k+1|k} [\overset{1}{p}_D \overset{1}{L}_{\mathbf{z}}] \cdot \overset{1}{\alpha}(\mathbf{z})}{\overset{1}{c}_{k+1}(\mathbf{z})} \quad (69)$$

$$\overset{2}{N}_{k+1|k+1} = \overset{2}{\alpha}_0 \overset{12}{S}_{k+1|k} [1 - \overset{2}{p}_D] + \sum_{\overset{2}{\mathbf{z}} \in \overset{2}{Z}_{k+1}} \frac{\overset{12}{S}_{k+1|k} [\overset{2}{p}_D \overset{2}{L}_{\mathbf{z}}] \cdot \overset{2}{\alpha}(\mathbf{z})}{\overset{2}{c}_{k+1}(\mathbf{z})} \quad (70)$$

Third, substituting Eqs. (59,60) into

$$\overset{1}{S}_{k+1|k+1}(\mathbf{x}) \cdot \overset{2}{S}_{k+1|k+1}(\mathbf{x}) \cdot \overset{12}{S}_{k+1|k}(\mathbf{x})^{-1} \quad (71)$$

yields

$$\begin{aligned} & \overset{1}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{2}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{12}{S}_{k+1|k}(\mathbf{x})^{-1} \\ &= \frac{\overset{1}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{2}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{12}{S}_{k+1|k}(\mathbf{x})}{\overset{1}{N}_{k+1|k+1} \cdot \overset{2}{N}_{k+1|k+1}} \end{aligned} \quad (72)$$

and thus

$$\sigma = \frac{\int \overset{1}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{2}{L}_{\mathbf{z}_{k+1}}(\mathbf{x}) \cdot \overset{12}{S}_{k+1|k}(\mathbf{x}) d\mathbf{x}}{\overset{1}{N}_{k+1|k+1} \cdot \overset{2}{N}_{k+1|k+1}} \quad (73)$$

$$= \frac{\overset{12}{S}_{k+1|k} [\overset{1}{L}_{\mathbf{z}_{k+1}} \overset{2}{L}_{\mathbf{z}_{k+1}}]}{\overset{1}{N}_{k+1|k+1} \cdot \overset{2}{N}_{k+1|k+1}} \quad (74)$$

Fourth, Eq. (44) becomes

$$\overset{12}{p}_{k+1|k+1}(n) = \frac{p(n) \cdot \sigma^n}{G(\sigma)} = \frac{\tilde{p}(n) \cdot \sigma^n}{\tilde{G}(\sigma)} \quad (75)$$

where $\tilde{p}(n)$ was defined in Eq. (56) and where

$$\tilde{G}(x) = \sum_{n \geq 0} \tilde{p}(n) \cdot x^n \quad (76)$$

This completes the derivation.

4.3 Derivation of PM-PHD filter equations

In this section I derive the multisensor PHD filter equations that were summarized in Section 2.2. I assume that the predicted target process and the clutter processes for the two sensors are all Poisson:

$$\overset{12}{p}_{k+1|k}(n) = e^{-\overset{12}{N}_{k+1|k}} \cdot \frac{\overset{12}{N}_{k+1|k}^n}{n!} \quad (77)$$

$$\overset{12}{G}_{k+1|k}(x) = \exp\left(-\overset{12}{N}_{k+1|k} + \overset{12}{N}_{k+1|k} \cdot x\right) \quad (78)$$

$$\overset{1}{C}_{k+1}(z) = \exp\left(-\overset{1}{\lambda}_{k+1|k} + \overset{1}{\lambda}_{k+1|k} \cdot z\right) \quad (79)$$

$$\overset{2}{C}_{k+1}(z) = \exp\left(-\overset{2}{\lambda}_{k+1|k} + \overset{2}{\lambda}_{k+1|k} \cdot z\right) \quad (80)$$

The derivation proceeds in a series of steps. These steps involve substituting Eqs. (77-80) into the PM-CPHD filter measurement-update equations, Eqs. (9-21).

First, Eq. (18) reduces to

$$\overset{1}{\alpha}_0 = \overset{12}{N}_{k+1|k}, \quad \overset{2}{\alpha}_0 = \overset{12}{N}_{k+1|k} \quad (81)$$

For,

$$\overset{j}{\alpha}_0 = \frac{\sum_{l=0}^j \overset{j}{C}_{k+1}^{(m-l)}(0) \cdot \overset{12}{G}_{k+1|k}^{(l+1)}(\gamma) \cdot \overset{j}{\sigma}_l(\mathbf{Z}_{k+1})}{\sum_{i=0}^j \overset{j}{C}_{k+1}^{(m-i)}(0) \cdot \overset{12}{G}_{k+1|k}^{(i)}(\gamma) \cdot \overset{j}{\sigma}_i(\mathbf{Z}_{k+1})} \quad (82)$$

$$\begin{aligned} &= \frac{\sum_{l=0}^j e^{-\overset{j}{\lambda}_{k+1}} \cdot \overset{j}{\lambda}_{k+1}^{m-l} \cdot e^{-\overset{12}{N}_{k+1|k} - \overset{12}{N}_{k+1|k} \gamma} \cdot \overset{12}{N}_{k+1|k}^{l+1} \cdot \overset{j}{\sigma}_l(\mathbf{Z}_{k+1})}{\sum_{i=0}^j e^{-\overset{j}{\lambda}_{k+1}} \cdot \overset{j}{\lambda}_{k+1}^{m-i} \cdot e^{-\overset{12}{N}_{k+1|k} - \overset{12}{N}_{k+1|k} \gamma} \cdot \overset{12}{N}_{k+1|k}^i \cdot \overset{j}{\sigma}_i(\mathbf{Z}_{k+1})} \end{aligned} \quad (83)$$

$$\begin{aligned} &= \overset{12}{N}_{k+1|k} \\ &\text{Second, Eq. (19) reduces to} \end{aligned} \quad (84)$$

$$\overset{j}{\alpha}(\mathbf{z}) = \overset{12}{N}_{k+1|k} \cdot \frac{\overset{j}{c}_{k+1}(\mathbf{z})}{\overset{j}{\lambda}_{k+1} \overset{j}{C}_{k+1}(\mathbf{z}) + \overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}}]} \quad (85)$$

For,

$$\overset{j}{\alpha}(\mathbf{z}) = \frac{\sum_{l=0}^{m-1} \overset{j}{C}_{k+1}^{(m-l-1)}(0) \cdot \overset{12}{G}_{k+1|k}^{(l+1)}(\gamma) \cdot \overset{j}{\sigma}_l(\mathbf{Z}_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^j \overset{j}{C}_{k+1}^{(m-i)}(0) \cdot \overset{12}{G}_{k+1|k}^{(i)}(\gamma) \cdot \overset{j}{\sigma}_i(\mathbf{Z}_{k+1})} \quad (86)$$

$$\begin{aligned} &= \frac{\sum_{l=0}^{m-1} e^{-\overset{j}{\lambda}_{k+1}} \cdot \overset{j}{\lambda}_{k+1}^{m-l-1} \cdot e^{-\overset{12}{N}_{k+1} + \overset{12}{N}_{k+1} \gamma} \cdot \overset{12}{N}_{k+1|k}^{l+1} \cdot \overset{j}{\sigma}_l(\mathbf{Z}_{k+1} - \{\mathbf{z}\})}{\sum_{i=0}^j e^{-\overset{j}{\lambda}_{k+1}} \cdot \overset{j}{\lambda}_{k+1}^{m-i} \cdot e^{-\overset{12}{N}_{k+1} + \overset{12}{N}_{k+1} \gamma} \cdot \overset{12}{N}_{k+1|k}^i \cdot \overset{j}{\sigma}_i(\mathbf{Z}_{k+1})} \end{aligned} \quad (87)$$

$$\begin{aligned} &= \frac{\overset{12}{N}_{k+1|k} \cdot \sum_{l=0}^{m-1} \overset{j}{\lambda}_{k+1}^{-l} \cdot \overset{12}{N}_{k+1|k}^l \cdot \overset{j}{\sigma}_l(\mathbf{Z}_{k+1} - \{\mathbf{z}\})}{\overset{j}{\lambda}_{k+1} \cdot \sum_{i=0}^j \overset{j}{\lambda}_{k+1}^{-i} \cdot \overset{12}{N}_{k+1|k}^i \cdot \overset{j}{\sigma}_i(\mathbf{Z}_{k+1})} \end{aligned} \quad (88)$$

Now note that

$$\overset{j}{\lambda}_{k+1}^{-i} \overset{12}{N}_{k+1|k}^i \overset{j}{\sigma}_i(\mathbf{Z}_{k+1}) = \overset{j}{\eta}_i(\mathbf{Z}_{k+1}) \quad (89)$$

where, using Eq. (20),

$$\overset{j}{\eta}_i(\mathbf{Z}_{k+1}) = \sigma_j \left(\frac{\overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}_1}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_1)}, \dots, \frac{\overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}_m}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_m)} \right) \quad (90)$$

Thus

$$\sum_{i=0}^j \overset{j}{\lambda}_{k+1}^{-i} \overset{12}{N}_{k+1|k}^i \overset{j}{\sigma}_i(\mathbf{Z}_{k+1}) = \sum_{i=0}^j \overset{j}{\eta}_i(\mathbf{Z}_{k+1}) \quad (91)$$

$$= \left(1 + \frac{\overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}_1}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_1)} \right) \dots \left(1 + \frac{\overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}_m}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_m)} \right) \quad (92)$$

$$= \prod_{i=1}^m \frac{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_i) + \overset{12}{D}_{k+1|k} [\overset{j}{p}_D \overset{j}{L}_{\mathbf{z}_i}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}_i)} \quad (93)$$

Likewise,

$$\sum_{l=0}^{m-1} \overset{j}{\lambda}_{k+1}^{-l} \overset{12}{N}_{k+1|k}^l \overset{j}{\sigma}_l(\mathbf{Z}_{k+1} - \{\mathbf{z}\}) = \sum_{l=0}^{m-1} \overset{j}{\eta}_l(\mathbf{Z}_{k+1} - \{\mathbf{z}\}) \quad (94)$$

$$= \prod_{\substack{i=1, \\ \mathbf{z}_i \neq \mathbf{z}}}^m \frac{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}_i) + D_{k+1|k} [p_D L_{\mathbf{z}_i}^j]}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z})} \quad (95)$$

Thus Eq. (88) becomes

$$\alpha(\mathbf{z}) = \frac{N_{k+1|k}^{12}}{\lambda_{k+1}^j} \cdot \frac{1}{\frac{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z})}} \quad (96)$$

$$= N_{k+1|k}^{12} \cdot \frac{c_{k+1}^j(\mathbf{z})}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]} \quad (97)$$

as claimed.

Third, it follows that Eq. (16) becomes

$$L_{\mathbf{z}_{k+1}}^j(\mathbf{x}) = \alpha_0(1 - p_D(\mathbf{x})) + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}^j(\mathbf{x}) \cdot \alpha(\mathbf{z})}{c_{k+1}^j(\mathbf{z})} \quad (98)$$

$$= N_{k+1|k}^{12} (1 - p_D(\mathbf{x})) + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) L_{\mathbf{z}}^j(\mathbf{x}) \cdot N_{k+1|k}^{12}}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]} \quad (99)$$

Because of the form of $N_{k+1|k+1}^j$ to be derived next, the quantity $N_{k+1|k}^{12}$ will cancel out in the numerator and denominator of

$$\frac{L_{\mathbf{z}_{k+1}}^j(\mathbf{x})}{N_{k+1|k+1}^j} \quad (100)$$

in Eqs. (11,12). Thus we can eliminate this factor and replace the original $L_{\mathbf{z}_{k+1}}^j(\mathbf{x})$ with

$$L_{\mathbf{z}_{k+1}}^j(\mathbf{x}) = 1 - p_D(\mathbf{x}) + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) L_{\mathbf{z}}^j(\mathbf{x})}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]} \quad (101)$$

Fourth, Eq. (17) becomes

$$v_{k+1|k+1}^j = s_{k+1|k}^{12} [1 - p_D] + \sum_{\mathbf{z} \in Z_{k+1}} \frac{s_{k+1|k}^{12} [p_D L_{\mathbf{z}}^j] \cdot \alpha(\mathbf{z})}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]} \quad (102)$$

For,

$$N_{k+1|k+1}^j = \alpha_0 s_{k+1|k}^{12} [1 - p_D] + \sum_{\mathbf{z} \in Z_{k+1}} \frac{s_{k+1|k}^{12} [p_D L_{\mathbf{z}}^j] \alpha(\mathbf{z})}{c_{k+1}^j(\mathbf{z})} \quad (103)$$

$$= N_{k+1|k}^{12} s_{k+1|k}^{12} [1 - p_D] + \sum_{\mathbf{z} \in Z_{k+1}} \frac{N_{k+1|k}^{12} \cdot s_{k+1|k}^{12} [p_D L_{\mathbf{z}}^j]}{\lambda_{k+1}^j c_{k+1}^j(\mathbf{z}) + D_{k+1|k} [p_D L_{\mathbf{z}}^j]} \quad (104)$$

$$= N_{k+1|k}^{12} \cdot v_{k+1|k}^j \quad (105)$$

Since $N_{k+1|k}^{12}$ will cancel out in the fraction in Eq. (100), we can replace $N_{k+1|k+1}^j$ with $v_{k+1|k+1}^j$.

Fifth, Eq. (15) becomes

$$\ell_{\mathbf{z}_{k+1}}^j(n) = \sum_{l=0}^m l! C_{n,l} \cdot \lambda_{k+1}^{m-l} \cdot \gamma^{n-l} \cdot \sigma_l(\mathbf{z}_{k+1}) \quad (106)$$

where

$$\gamma = s_{k+1|k}^{12} [1 - p_D] \quad (107)$$

(where, note, $C_{n,l} = 0$ if $l > n$ or $l < 0$). For,

$$\ell_{\mathbf{z}_{k+1}}^j(n) = \sum_{l=0}^{\min\{n,m\}} C_{k+1}^{(m-l)}(0) \cdot l! C_{n,l} \cdot \gamma^{n-l} \cdot \sigma_l(\mathbf{z}_{k+1}) \quad (108)$$

$$= \sum_{l=0}^m e^{-\lambda_{k+1}^j} \cdot \lambda_{k+1}^{m-l} \cdot l! C_{n,l} \cdot \gamma^{n-l} \cdot \sigma_l(\mathbf{z}_{k+1}) \quad (109)$$

Constant factors will cancel out in the numerator and denominator of

$$\frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \quad (110)$$

of Eq. (11). So we can eliminate the factor $e^{-\lambda_{k+1}^j}$ to get

$$\ell_{\mathbf{z}_{k+1}}^j(n) = \sum_{l=0}^m \lambda_{k+1}^{m-l} \cdot l! C_{n,l} \cdot \gamma^{n-l} \cdot \sigma_l(\mathbf{z}_{k+1}) \quad (111)$$

and so the right-hand side can replace the previous definition in Eq. (15) and we get, as claimed, Eq. (26).

Sixth, Eq. (14) becomes

$$\tilde{p}(n) = \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot p_{k+1|k}^{1..s}(n) \quad (112)$$

$$= \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot e^{-N_{k+1|k}^{12}} \cdot \frac{N_{k+1|k}^{12 n}}{n!} \quad (113)$$

and so Eq. (13) becomes

$$\tilde{G}(x) = e^{-N_{k+1|k}^{12}} \sum_{n \geq 0} \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^n}{n!} \quad (114)$$

which, because of Eq. (110), we can replace with

$$\tilde{G}(x) = \sum_{n \geq 0} \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^n}{n!} \quad (115)$$

Seventh,

$$\tilde{G}^{(1)}(\sigma) = N_{k+1|k}^{12} \sum_{j \geq 0} \ell_{\mathbf{z}_{k+1}}^1(j+1) \cdots \ell_{\mathbf{z}_{k+1}}^s(j+1) \cdot \frac{(N_{k+1|k}^{12} \cdot \sigma)^j}{j!} \quad (116)$$

For,

$$\tilde{G}^{(1)}(x) = \sum_{n \geq 1} \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot n N_{k+1|k}^{12} \cdot \frac{(N_{k+1|k}^{12} \cdot x)^{n-1}}{n!} \quad (117)$$

$$= N_{k+1|k}^{12} \sum_{n \geq 1} \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^{n-1}}{(n-1)!} \quad (118)$$

$$= N_{k+1|k}^{12} \sum_{j \geq 0} \ell_{\mathbf{z}_{k+1}}^1(j+1) \cdots \ell_{\mathbf{z}_{k+1}}^s(j+1) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^j}{j!} \quad (119)$$

Eighth, it follows that

$$\frac{1}{N_{k+1|k}^{12}} \frac{G^{(1)}(\sigma)}{G(\sigma)} = \phi \quad (120)$$

$$= \frac{\sum_{j \geq 0} \ell_{\mathbf{z}_{k+1}}^1(j+1) \cdots \ell_{\mathbf{z}_{k+1}}^s(j+1) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^j}{j!}}{\sum_{n \geq 0} \ell_{\mathbf{z}_{k+1}}^1(n) \cdots \ell_{\mathbf{z}_{k+1}}^s(n) \cdot \frac{(N_{k+1|k}^{12} \cdot x)^n}{n!}}$$

where ϕ was defined in Eq. (25).

Ninth, from Eq. (9) and Eq. (11),

$$D_{k+1|k+1}^{12}(\mathbf{x}) = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \cdot \frac{L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x})}{N_{k+1|k+1}^1 \cdots N_{k+1|k+1}^s} \cdot S_{k+1|k}^{12}(\mathbf{x}) \quad (121)$$

$$= \frac{1}{N_{k+1|k}^{12}} \cdot \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \cdot \frac{L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x})}{N_{k+1|k+1}^1 \cdots N_{k+1|k+1}^s} \cdot D_{k+1|k}^{12}(\mathbf{x}) \quad (122)$$

$$= \phi \cdot \frac{L_{k+1}^1(\mathbf{x}) \cdots L_{k+1}^s(\mathbf{x})}{N_{k+1|k+1}^1 \cdots N_{k+1|k+1}^s} \cdot D_{k+1|k}^{12}(\mathbf{x}) \quad (123)$$

5 Conclusion

It is possible to derive measurement-update equations for a “true” multisensor PHD filter and a “true” multisensor CPHD filter, but these equations are computationally intractable in general. The usual heuristic “fixes” for this problem, the iterated-corrector PHD filter and the iterated-corrector CPHD filter, are not entirely satisfactory from a theoretical point of view.

In this paper I have shown that there a middle ground between (on the one hand) the “true” but intractable multisensor filters and (on the other) the tractable but heuristic iterated-corrector filters. I call these two new filters the multisensor PHD (PM-PHD) and multisensor CPHD (PM-CPHD) filters.

The PM-CPHD filter is potentially computationally tractable, provided that we assume that the cardinality distributions $p_{k+1|k}^{1..s}(n)$ are vanishing for larger values of n . Given this, the computational complexity of the PM-CPHD filter appears to be $O(m^1 \cdots m^s \cdot n)$, where n is the current number of targets and m^1, \dots, m^s are the current numbers of measurements collected by the s sensors.

The PM-PHD filter may be more computationally problematic. In this case the $p_{k+1|k}^{1..s}(n)$ are not necessarily vanishing for larger of values of n . As a result, the numerator and denominator of Eq. (25) become infinite sums. If it turns out to be impossible to approximate these sums, the PM-PHD filter may turn out to be impractical.

Further research is required to resolve such matters. Implementation and simulation results will be the subject of some future paper.

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